

The arctangent law for a certain random time related to a one-dimensional diffusion

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Abstract

For a time-homogeneous, one-dimensional diffusion process $X(t)$, we investigate the distribution of the first instant, after a given time r , at which $X(t)$ exceeds its maximum on the interval $[0, r]$, generalizing a result of Papanicolaou, which is valid for Brownian motion.

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1. Introduction

In this paper, we extend to a one-dimensional diffusion process $X(t)$ the result of (Papanicolaou, 2016) for Brownian motion, concerning the arctangent law for a certain random time.

Indeed, let be $X(t)$ a time-homogeneous, one-dimensional diffusion in the interval $I \subset \mathbb{R}$ which is the solution of the SDE:

$$dX(t) = \mu(X(t))dt + \sigma(X(t))dB_t, \quad X(0) = \eta \in I, \quad (1.1)$$

where B_t (with $B_0 = 0$) is standard Brownian motion (BM) and the drift and diffusion coefficients satisfy the usual conditions (see e.g. (Ikeda and Watanabe, 1981)) for existence and uniqueness of the solution of (1.1).

For a fixed time $r > 0$, we consider the maximum $M_r := \max_{0 \leq t \leq r} X(t)$ of the diffusion X on the interval $[0, r]$, and we denote by S the following random time:

$$S = S(r) := \inf\{t \geq r : X(t) \geq M_r\} - r. \quad (1.2)$$

Assuming that the initial state η is random, our aim is to study the distribution function of S , generalizing the result of (Papanicolaou, 2016), that refers to the case when $X(t) = B_t^\eta := \eta + B_t$ (i.e. BM starting from the random value η , not necessarily zero), and states that:

$$P\{S_B(r) \leq t\} = \frac{2}{\pi} \arctan \left(\sqrt{\frac{t}{r}} \right), \quad t \geq 0, \quad (1.3)$$

where $S_B(r) = \inf\{t \geq r : B_t^\eta \geq \max_{s \in [0, r]} B_s^\eta\} - r$. By taking the derivative with respect to t , one obtains the probability density of $S_B(r)$:

$$f_{S_B(r)}(t) = \frac{\sqrt{r}}{\pi(r+t)\sqrt{t}}, \quad t \geq 0. \quad (1.4)$$

Notice that the expectation, $E(S_B(r))$, turns out to be infinite.

The knowledge of the distribution of S is relevant in various diffusion models used in applied sciences, such as Mathematical Finance, Biology, Physics, Hydraulics, etc., whenever the time evolution of the phenomenon under study is described by a diffusion $X(t)$; in fact, one is often interested to find the first instant, after a given time r , at which $X(t)$ exceeds the maximum value attained in the time interval $[0, r]$, namely in times prior to r . For instance, in the Economy framework, if we let r vary in $(0, +\infty)$, the process $S(r)$, so obtained, is related to the drawdown process, which measures the fall in value of $X(t)$ from its running maxima, and is frequently used as performance indicator in the fund management industry (see e.g. (Dassios and Lim, 2017) and references therein). Indeed, $S(r)$ can be expressed in terms of the time elapsed since the last time the maximum is achieved, that was studied in (Dassios and Lim, 2017).

2. The result

Let $w(x) \in C^2(I)$ be the *scale function* associated to the diffusion $X(t)$ driven by the SDE (1.1), that is, the solution of:

$$\begin{cases} Lw(x) = 0, & x \in I \\ w(0) = 0, & w'(0) = 1, \end{cases} \quad (2.1)$$

where L is the infinitesimal generator of X defined by:

$$Lh = \frac{1}{2}\sigma^2(x)\frac{d^2h}{dx^2} + \mu(x)\frac{dh}{dx}, \quad h \in C^2(I). \quad (2.2)$$

Actually, the scale function can be taken as any function $\tilde{w} = aw + b$, with $a > 0$ and $b \in \mathbb{R}$ (see e.g. (Karlin and Taylor, 1975)); we chose the initial conditions of (2.1), for the sake of simplicity.

As easily seen, if the integral $\int_0^t \frac{2\mu(z)}{\sigma^2(z)} dz$ converges, the problem (2.1) has solution:

$$w(x) = \int_0^x \exp\left(-\int_0^t \frac{2\mu(z)}{\sigma^2(z)} dz\right) dt. \quad (2.3)$$

If $\zeta(t) := w(X(t))$, by Itô's formula one obtains

$$\zeta(t) = w(\eta) + \int_0^t w'(w^{-1}(\zeta(s)))\sigma(w^{-1}(\zeta(s)))dB_s, \quad (2.4)$$

that is, the process $\zeta(t)$ is a local martingale, whose quadratic variation is

$$\rho(t) \doteq \langle \zeta \rangle_t = \int_0^t [w'(X(s))\sigma(X(s))]^2 ds, \quad t \geq 0. \quad (2.5)$$

The (random) function $\rho(t)$ is differentiable, increasing, and $\rho(0) = 0$. If $\rho(+\infty) = +\infty$, it can be shown (see e.g. (Revuz and Yor, 1991)) that there exists a BM \widehat{B} such that $\zeta(t) = \widehat{B}(\rho(t)) + w(\eta)$; thus, since w is invertible, the solution $X(t)$ to (1.1) can be written in the form

$$X(t) = w^{-1}(\widehat{B}(\rho(t)) + w(\eta)). \quad (2.6)$$

In this way, X is obtained from BM by a space transformation and a random time-change (see e.g. the discussion in (Abundo, 2012)).

Definition 2.1. We say that the diffusion $X(t)$ (with $X(0) = x$) is *conjugated* to BM (see also (Abundo, 2012)), if there exists an increasing differentiable function $v(x)$ with $v(0) = 0$, such that $X(t) = v^{-1}(B_t + v(x))$, for any $t \geq 0$.

Remark 2.2. Diffusions conjugated to BM are special cases of (2.6), for $\rho(t) = t$, $\widehat{B}_t = B_t$, and $w = v$ (however, it is not required that $v'(0) = 1$).

A class of diffusions conjugated to BM is given by processes $X(t)$ which are solutions of SDEs such as:

$$dX(t) = \frac{1}{2}\sigma(X(t))\sigma'(X(t))dt + \sigma(X(t))dB_t, \quad X(0) = x, \quad (2.7)$$

with $\sigma(\cdot) \geq 0$. Indeed, if the integral $v(x) := \int^x \frac{1}{\sigma(r)}dr$ is convergent, by Itô's formula, one obtains $X(t) = v^{-1}(B_t + v(x))$.

Explicit examples of diffusions conjugated to BM (see also (Abundo, 2012)) are:

- the diffusion in $I = \mathbb{R}$ driven by the SDE $dX(t) = \frac{1}{3}X(t)^{1/3}dt + X(t)^{2/3}dB_t$, $X(0) = x$, which is conjugated to BM via the function $v(x) = 3x^{1/3}$, that is, $X(t) = (x^{1/3} + \frac{1}{3}B_t)^3$;
- the diffusion in $I = [0, +\infty)$ driven by the SDE $dX(t) = \frac{1}{4}dt + \sqrt{X(t) \vee 0} dB_t$, $X(0) = x \geq 0$ (Feller process), which is conjugated to BM via the function $v(x) = 2\sqrt{x}$, that is, $X(t) = \frac{1}{4}(B_t + 2\sqrt{x})^2$;
- the diffusion in $I = [0, 1]$ driven by the SDE $dX(t) = (\frac{1}{4} - \frac{1}{2}X(t))dt + \sqrt{X(t)(1 - X(t)) \vee 0} dB_t$, $X(0) = x \in [0, 1]$ (Wright-Fisher like process), which is conjugated to BM via the function $v(x) = 2 \arcsin \sqrt{x}$, that is, $X(t) = \sin^2(B_t/2 + \arcsin \sqrt{x})$.

If we drop the requirement that $v(0) = 0$ in Definition 2.1, then, for $\sigma > 0$, the diffusion in $I = (0, +\infty)$ driven by the SDE $dX(t) = \frac{\sigma^2}{2}X(t)dt + \sigma X(t)dB_t$, $X(0) = x > 0$ (a special case of geometric BM) is conjugated to BM via the function $v(x) = \frac{\ln x}{\sigma}$, that is, $X(t) = \exp(\sigma B_t + \ln x)$.

The class of processes X given by (2.6) with $\rho(t)$ deterministic, includes, besides diffusions conjugated to BM, the integral of Gauss-Markov processes (see (Abundo, 2015), (Abundo, 2013)), e.g. integrated BM $X(t) = \int_0^t B_s ds$, represented by $X(t) = \widehat{B}(\rho(t))$ with $\rho(t) = t^3/3$, and integrated Ornstein-Uhlenbeck (OU) process $X(t) = \int_0^t Y(s)ds$, where $Y(t)$ is OU process (see (Abundo, 2013) for the explicit representation of X in the form (2.6)).

The announced result is:

Theorem 2.3. *Let $X(t)$ be the solution of the SDE (1.1) and suppose that the scale function w of X , given by (2.1) exists; for fixed $r > 0$, let $S(r)$ be the random time defined by (1.2). With the previous notations, suppose that the function ρ satisfies the condition $\rho(+\infty) = +\infty$;*

(i) if $\rho(t)$ is deterministic, then the probability distribution of $S(r)$ is:

$$P\{S(r) \leq t\} = \frac{2}{\pi} \arctan \left(\sqrt{\frac{\rho(t+r) - \rho(r)}{\rho(r)}} \right), \quad t \geq 0, \quad (2.8)$$

and its density is:

$$f_{S(r)}(t) = \frac{\rho'(t+r)\sqrt{\rho(r)}}{\rho(t+r)\sqrt{\rho(t+r)-\rho(r)}}. \quad (2.9)$$

(ii) If ρ is not deterministic, let us suppose that there exist two deterministic, continuous increasing functions $\alpha(t)$ and $\beta(t)$, with $\alpha(0) = \beta(0) = 0$, such that

$$\alpha(t) \leq \rho(t) \leq \beta(t), \quad \forall t \geq 0. \quad (2.10)$$

Then:

$$P\{S(r) \leq t\} \leq \frac{2}{\pi} \arctan \left(\sqrt{\frac{\beta(t+r) - \alpha(r)}{\alpha(r)}} \right), \quad t \geq 0. \quad (2.11)$$

Moreover, if there exists $\bar{t} > 0$ such that $\alpha(r + \bar{t}) \geq \beta(r)$, then:

$$P\{S(r) \leq t\} \geq \frac{2}{\pi} \arctan \left(\sqrt{\frac{\alpha(t+r) - \beta(r)}{\beta(r)}} \right), \quad t > \bar{t}. \quad (2.12)$$

Proof. Under the hypothesis, the representation (2.6) of X holds.

(i) Suppose that $\rho(t)$ is deterministic. Then:

$$M_r = \max_{t \in [0, r]} X(t) = \max_{t \in [0, r]} w^{-1} \left(\widehat{B}(\rho(t)) + w(\eta) \right) = w^{-1} \left(\max_{t \in [0, \rho(r)]} \widehat{B}_t + w(\eta) \right),$$

and so:

$$\tau_r := \inf \left\{ t \geq r : X(t) \geq M_r \right\} = \inf \left\{ t \geq r : \widehat{B}(\rho(t)) \geq \max_{u \in [0, \rho(r)]} \widehat{B}_u \right\}.$$

Thus, by recalling the definition of $S_B(r)$, and taking \widehat{B} in place of B and $\rho(r)$ in place of r , we get:

$$\rho(\tau_r) = \inf \left\{ s \geq \rho(r) : \widehat{B}_s \geq M_{\rho(r)}^{\widehat{B}} \right\} = S_{\widehat{B}}(\rho(r)) + \rho(r), \quad (2.13)$$

where $M_r^{\widehat{B}} = \max_{u \in [0, r]} \widehat{B}_u$. Therefore, $\tau_r = \rho^{-1} (S_{\widehat{B}}(\rho(r)) + \rho(r))$, and $S(r) = \tau_r - r = \rho^{-1} (S_{\widehat{B}}(\rho(r)) + \rho(r)) - r$. Finally:

$$\begin{aligned} P\{S(r) \leq t\} &= P\{\rho^{-1}(S_{\widehat{B}}(\rho(r)) + \rho(r)) \leq t + r\} \\ &= P\{S_{\widehat{B}}(\rho(r)) + \rho(r) \leq \rho(t + r)\} = P\{S_{\widehat{B}}(\rho(r)) \leq \rho(t + r) - \rho(r)\}, \end{aligned} \quad (2.14)$$

from which (2.8) follows, by using (1.3); formula (2.9) is obtained by taking the derivative with respect to t .

(ii) Suppose that $\rho(t)$ is not deterministic, and the bounds (2.10) hold; set:

$$\tau_{r,\alpha} = \inf\{t \geq r : \widehat{B}(\rho(t)) \geq \max_{u \in [0, \alpha(r)]} \widehat{B}_u\}, \quad \tau_{r,\beta} = \inf\{t \geq r : \widehat{B}(\rho(t)) \geq \max_{u \in [0, \beta(r)]} \widehat{B}_u\}. \quad (2.15)$$

As easily seen, one has:

$$\tau_{r,\alpha} \leq \tau_r \leq \tau_{r,\beta}, \quad (2.16)$$

that implies:

$$\rho(\tau_{r,\alpha}) \leq \rho(\tau_r) \leq \rho(\tau_{r,\beta}). \quad (2.17)$$

Moreover, since $\rho(\tau_{r,\alpha}) = \inf\{s > \rho(r) : \widehat{B}_s \geq \max_{u \in [0, \alpha(r)]} \widehat{B}_u\}$, and $\rho(\tau_{r,\beta}) = \inf\{s > \rho(r) : \widehat{B}_s \geq \max_{u \in [0, \beta(r)]} \widehat{B}_u\}$, we have :

$$\inf\{s > \alpha(r) : \widehat{B}_s \geq \max_{u \in [0, \alpha(r)]} \widehat{B}_u\} \leq \rho(\tau_{r,\alpha}) \quad (2.18)$$

and

$$\rho(\tau_{r,\beta}) \leq \inf\{s > \beta(r) : \widehat{B}_s \geq \max_{u \in [0, \beta(r)]} \widehat{B}_u\}. \quad (2.19)$$

Thus, recalling the definition of $S_{\widehat{B}}(r)$, from (2.17) we get:

$$S_{\widehat{B}}(\alpha(r)) + \alpha(r) \leq \rho(\tau_r) \leq S_{\widehat{B}}(\beta(r)) + \beta(r), \quad (2.20)$$

and so:

$$\rho^{-1}(S_{\widehat{B}}(\alpha(r)) + \alpha(r)) \leq \tau_r \leq \rho^{-1}(S_{\widehat{B}}(\beta(r)) + \beta(r)), \quad (2.21)$$

where ρ^{-1} is the “inverse” of the random function ρ , which is defined by $\rho^{-1}(s) := \inf\{t > 0 : \rho(t) > s\}$. Since (2.10) implies that $\beta^{-1}(s) \leq \rho^{-1}(s) \leq \alpha^{-1}(s)$, we obtain:

$$\beta^{-1}(S_{\widehat{B}}(\alpha(r)) + \alpha(r)) \leq \tau_r \leq \alpha^{-1}(S_{\widehat{B}}(\beta(r)) + \beta(r)). \quad (2.22)$$

Therefore:

$$\beta^{-1}(S_{\widehat{B}}(\alpha(r)) + \alpha(r)) - r \leq S(r) \leq \alpha^{-1}(S_{\widehat{B}}(\beta(r)) + \beta(r)) - r. \quad (2.23)$$

From the first inequality in (2.23), it follows that:

$$P\{S(r) \leq t\} \leq P\{S_{\widehat{B}}(\alpha(r)) \leq \beta(t+r) - \alpha(r)\} = \frac{2}{\pi} \arctan \left(\sqrt{\frac{\beta(t+r) - \alpha(r)}{\alpha(r)}} \right), \quad (2.24)$$

that proves (2.11). Moreover, if there exists $\bar{t} > 0$ such that $\alpha(r + \bar{t}) \geq \beta(r)$, we have $\alpha(r + t) \geq \beta(r)$ for $t > \bar{t}$, because $\alpha(t)$ is increasing; then, for $t > \bar{t}$ the second inequality in (2.23) implies:

$$P\{S(r) \leq t\} \geq P\{S_{\widehat{B}}(\beta(r)) \leq \alpha(t+r) - \beta(r)\} = \frac{2}{\pi} \arctan \left(\sqrt{\frac{\alpha(t+r) - \beta(r)}{\alpha(r)}} \right), \quad (2.25)$$

which proves (2.12). The condition $\alpha(t+r) \geq \beta(r)$ is necessary so that $S_{\widehat{B}}(\beta(r)) \geq 0$; of course, if a value $\bar{t} > 0$, such that $\alpha(r + \bar{t}) \geq \beta(r)$, does not exist, the inequality (2.12) loses meaning, because the square root is not defined. \square

Remark 2.4. Notice that (2.8) is independent of the scale function w ; in particular, if X is conjugated to BM via the function v , being $\rho(t) = t$, one obtains that the distribution of $S(r)$ is the same as that of $S_B(r)$, given by (1.3).

For an example of diffusion X for which ρ is not deterministic, but satisfies the bounds (2.10) with $\alpha(t)$ close to $\beta(t)$, see Example 4 of (Abundo, 2012).

Remark 2.5. Let us suppose that $\rho(t)$ is deterministic and there exists $\gamma > 0$ such that $\rho(t) \sim \text{const} \cdot t^\gamma$, as $t \rightarrow +\infty$; then, from (2.9) it easily follows that $E(S(r)) < +\infty$, provided that $\gamma > 2$. This is not the case of BM, because $\rho(t) = t$; instead, it holds e.g. for integrated BM $X(t) = \int_0^t B_s ds$, being $X(t) = \widehat{B}(\rho(t))$ with $\rho(t) = t^3/3$ (see (Abundo, 2015), (Abundo, 2013)).

Remark 2.6. Let us suppose that $\rho(t)$ is deterministic; for given r_1, r_2 with $0 \leq r_1 < r_2$, set $M_{[r_1, r_2]} := \max_{t \in [r_1, r_2]} X(t)$, and $S(r_1, r_2) = \inf\{t \geq r_2 : X(t) = M_{[r_1, r_2]}\} - r_2$. Then, by using the arguments of the proof of Theorem 2.3 and the Remark of (Papanicolaou, 2016) which refers to BM, one obtains:

$$P\{S(r_1, r_2) \leq t\} = \frac{2}{\pi} \arctan \left(\sqrt{\frac{\rho(t + r_2) - \rho(r_2)}{\rho(r_2) - \rho(r_1)}} \right), \quad t \geq 0. \quad (2.26)$$

Remark 2.7. Let us suppose that $\rho(t)$ is deterministic, and for $r > 0$, define $L(r) = \min_{t \in [0, r]} X(t)$, and $U(r) = \inf\{t \geq r : X(t) \geq L(r)\} - r$. Then, by using the fact, proved in (Papanicolaou, 2016), that $S_B(r)$ and $U_B(r) := \inf\{t \geq r : B_t^\eta \geq L_B(r)\} - r$ have the same distribution (here $L_B(r) = \min_{t \in [0, r]} B_t^\eta$), and by arguments analogous to those in the proof of Theorem 2.3, we conclude that also $U(r)$ and $S(r)$ have a common distribution.

3. Conclusions and final remarks

We have considered a time-homogeneous one-dimensional diffusion process X in an interval $I \subset \mathbb{R}$, driven by the SDE (1.1); for a given time r , under suitable conditions, we have found the distribution of the time $S(r)$ required (after r) for $X(t)$ to exceed its maximum $M_r = \max_{t \in [0, r]} X(t)$ on the interval $[0, r]$, generalizing the result in (Papanicolaou, 2016), which refers to BM. Indeed, we have reduced X to BM (see (2.6)) by a space transformation, given by the scale function $w(x)$, and a random time-change $\rho(t)$, under the assumption that $\rho(+\infty) = +\infty$ ($\rho(t)$ is the quadratic variation of the space-transformed process). Thus, we have shown that, when $\rho(t)$ is deterministic, $S(r)$ follows a compound arctangent law; note that, in this case $X(t)$ solves also the SDE:

$$dX(t) = -\frac{\rho'(t)w''(X(t))}{2(w'(X(t)))^3}dt + \frac{\sqrt{\rho'(t)}}{w'(X(t))}d\tilde{B}_t, \quad (3.1)$$

where $w'(x)$ and $w''(x)$ denote first and second derivative of $w(x)$, and \tilde{B} is BM (see also (Abundo, 2017)). The class of processes X , for which the distribution of $S(r)$ has been found, includes, besides diffusions conjugated to BM (see e.g. (Abundo, 2012)), integrated BM and integrated Ornstein-Uhlenbeck process (see (Abundo, 2015), (Abundo, 2013)).

As a curiosity, we note that a number of results are known, which regard inverse trigonometric laws for some random times associated to BM; for instance, the density (1.4) appears as the conditional density of the second inter-passage time of BM through a level, with the condition that the first-passage time is r (see eq. (2.15) of (Abundo, 2016)).

The arc-sine law is valid for the time τ spent by BM on the positive half-line during the time interval $[0, r]$, that is, $P(\tau \leq t) = \frac{2}{\pi} \arcsin \left(\sqrt{\frac{t}{r}} \right)$, $t \in [0, r]$ (see (Levy, 1965)); moreover, a compound arc-sine law holds for the first instant θ at which a diffusion X of the form (2.6), with $\rho(t)$ deterministic, attains the maximum in the interval $[0, r]$, namely $P(\theta \leq t) = \frac{2}{\pi} \arcsin \left(\sqrt{\frac{\rho(t)}{\rho(r)}} \right)$, $t \in [0, r]$ (see (Abundo, 2006), and (Levy, 1965) in the case of BM, i.e. $\rho(t) = t$).

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